REDUCING THE INCIDENCE OF MATHEMATICAL MISCONCEPTIONS IN 'MIDDLE BAND' STUDENTS

Philip Swedosh
The University of Melbourne
<swedosh@ms.unimelb.edu.au>

Previous studies conducted collaboratively by the author have shown that a strategy based on Piaget's notion of cognitive conflict was very successful in causing a reduction in the frequency of mathematical misconceptions exhibited by a group of very bright tertiary students and that much of this improvement persisted over time. This study investigates whether the same method effectively reduces the frequency of mathematical misconceptions exhibited by average or 'middle band' first year university students.

INTRODUCTION

Several researchers (Bell, 1982; Davis, 1984; Farrell, 1992; Margulies, 1993; Perso, 1992; Swedosh, 1996) have studied various aspects relating to the mathematical misconceptions exhibited. These include the variety of misconceptions, their frequencies of occurrence, and their importance to the student's future learning of mathematics. The number and range of different types of mathematical misconceptions is enormous, "and a complete list may not even be practical" (Davis, 1984, p. 335). Swedosh (1996) discussed the types, frequencies, and possible reasons for the misconceptions commonly exhibited by mathematics students on entering tertiary mathematics subjects at the University of Melbourne (U. of M.) and at LaTrobe University (LaTrobe).

The importance of understanding basic concepts, especially if continuing one's study of mathematics, is well documented. Eliminating misconceptions, where students have developed them, is therefore also of great importance. Government bodies have recognised the need for a sound preparation for further studies and for increasing the participation rate in post-secondary mathematics education (Ministry of Education Victoria, 1984; Victorian Government, 1987; Australian Education Council, 1990). In A National Statement on Mathematics for Australian Schools (Australian Education Council, 1990), a major goal is that "as a result of learning mathematics in school, all students should possess sufficient command of mathematical expressions, representations and technology to continue to learn mathematics independently and collaboratively" (p. 18).

In several areas of mathematics, success at a particular level depends heavily on previous mastery of basic concepts and also on being able to confidently use certain skills (Swedosh, 1996). Swedosh also argues that the level of understanding of some of the basic mathematical concepts which are expected to be acquired in secondary school determine, to a large extent, the preparedness of students to study tertiary mathematics. As the mastery of pre-requisite knowledge has such a profound influence on subsequent learning of mathematics, there has been great interest about mathematical misconceptions, the frequencies with which they occur, and any method which has the potential to reduce these frequencies, amongst a large number of mathematics teachers, lecturers, and tutors.

Preparedness of students to undertake tertiary mathematics is an issue of great significance in the light of statements made by Blyth and Calegari (1985) that "contrary to a widely held belief, 90% of all HSC students do apply for tertiary entrance. It is reasonable to infer from this that most students see HSC as a preparation for tertiary studies" (p. 312) and also, "statistics collected by the Mathematical Association of Victoria (MAV) show that 75% of all tertiary courses require a pass in HSC mathematics" (p. 312). These statements were made some years ago but there is evidence to suggest that this situation has not changed dramatically since then.

Having made pronouncements about the influence of misconceptions to future learning, several researchers (Bell, 1982; Davis, 1984; Farrell, 1992; Margulies, 1993; Perso, 1992; Swedosh, 1996) have studied ways to reduce or eliminate misconceptions. Many of them believe that teaching techniques can be developed which aim at diagnosing and eliminating those misconceptions by considering the misconceptions exhibited by students. A variety of attempts have been made to eliminate students' misconceptions in a diverse range of areas. Of these, the "conflict teaching approach", based on Piaget's notion of cognitive conflict, has met with some success. In this approach, a teacher and a learner discuss inconsistencies in the learner's thinking so that the learner is able to realise that the conceptions exhibited were inadequate or faulty and needed modification (Tirosh, 1990). Vinner (1990) supports this premise and states that "there is no doubt that if inconsistencies in the students' thinking are drawn to their attention, it will help some of them to resolve some inconsistencies in a desirable way" (p. 97). Several authors have found the conflict teaching approach to be an effective method to successfully resolve a range of misconceptions relating to aspects of mathematics and physics (Stavy and Berkovitz, 1980; Strauss, 1972; Swan, 1983; Swedosh and Clark, 1997).

Swedosh and Clark (1997) used the conflict teaching strategy in a first year mathematics class at the U. of M which comprised very bright and mathematically strong students (as indicated by their very high Year 12 scores) and found that the frequency of mathematical misconceptions was greatly reduced after the strategy had been employed.

> It is clear that by first challenging or undermining the misconception held by the students by showing the ridiculous outcomes which can flow from such 'rules', and then replacing the 'damaged' concept with the correct one, mathematical misconceptions can be, to a great extent, eliminated (p. 498).

The study by Swedosh and Clark (1997) found the strategy to be effective in significantly reducing mathematical misconceptions. Despite this success, several questions remained including whether the improvement seen would persist, and whether the method would be as effective with students who were not as able. There was some concern that the improvement might be a short term phenomenon and that students might revert to the conceptions which they previously held. Swedosh and Clark (1998) found that

> The results show that while a small proportion of the improvement diminished, a large improvement was still evident one year later and most of the benefit to students had been retained. (p. 595).

There was also concern that the results may not be replicated with students who were not as able as the original group. Swedosh and Clark (1998) stated that

> The improvement for students whose backgrounds were not as good may not be as dramatic or may not occur. The reason for this conjecture is two-fold: the students involved in this study were capable of recognising the inconsistencies in their previous thinking and then learning the correct concepts quickly; and these students demonstrated during discussions with the authors that they were embarrassed by making the errors that they did on what they considered to be material of such an elementary nature, and therefore had a strong desire to remedy the situation. Both of these qualities are likely to be more pronounced with better students (p. 594).

The goal in this study is to ascertain how generalisable the conflict teaching approach is in terms of its efficacy with students of different ability levels. Having determined that this strategy is extremely effective with very able students, and that the benefit persists over time, it is important to learn whether the strategy can be successfully employed to significantly decrease the frequency of misconceptions exhibited by average students thereby improving their chances of future success in their study of mathematics.

METHODOLOGY

The experiment was set up with a treatment group who would be subjected to the conflict teaching approach and a control group who would not. Each of the groups was comprised of students who were studying the same first year university mathematics subject at the U. of M. in Semester 1, 1998. The subject, which is known as Introductory Mathematics, is the easiest first year university mathematics subject which is offered at the U. of M. Both of the groups were taught by a lecturer who had consistently received high ratings from the students for their teaching.

It was considered important that the backgrounds of the students be comparable so that the results of this study could be put into context and not be distorted by inherent differences between the groups. The students reported on in this study are those students who sat both of the tests, and who had completed Mathematics Methods 3/4 (MM) but not Specialist Mathematics as part of their Victorian Certificate of Education (V.C.E.) in 1997 prior to enrolling at the U. of M. in 1998. There were 90 students in this category in the treatment group and 50 students in this category in the control group. All students who did not fit into this category have been excluded from this study. Those excluded from this study include the few students who studied Introductory Mathematics having completed Specialist Mathematics (the most difficult V.C.E. mathematics subject), mature age students, as well as interstate and overseas students each of whose backgrounds were quite different to the students considered herein.

In each of the three Common Assessment Tasks (CATs) in a V.C.E. subject, students are awarded a grade from E to A+ corresponding to a ten point scale from one to ten. Using this scale, Table 1 shows the average mark for each group for MM CAT 1 (the challenging problem), for MM CAT 2 (facts and skills), for MM CAT 3 (the analysis task). Table 1 also shows the average Tertiary Entrance Ranking (a percentile with the highest possible ranking of 99.95 and with about 23 students for each .05 — .05% of students in the state is about 23) for each group.

Table 1
Average Scores for the Two Groups

Students in Each Group	TER	Average Score Maths Methods CAT 1	CAT 2	CAT 3
90 students in the treatment group:	82.82	8.26	7.53	7.20
50 students in the control group:	79.00	7.84	6.94	6.56

There appears to be a small difference in the background levels of the two groups, but, as the statistical analysis to be used is one which compares the relative extent of improvement in each group, the difference is not large enough to cause concern and the two groups can be considered to have similar backgrounds.

Both groups were taught the same material at the same time and both groups were given the same tests at the same times. An internal control mechanism was also set up by having some questions on the test which neither group received any treatment. The objective was to control extraneous influences so that if a significant difference was found between the performances of students in the two groups, we would have eliminated the effect of the teacher (as much as is practicable), the effect of being taught the material, and the effect of the testing, on the improvement made. We would therefore be able to attribute any difference in the improvement of the two groups to the treatment.

The test consisted of nine questions and was a subset of the seventeen questions in the test used in Swedosh and Clark (1997). These nine questions were selected as they were considered to be at an appropriate level for these students.

The original seventeen questions consisted of questions which were similar to the questions posed in earlier tests at the U. of M. and at LaTrobe (Worley, 1993) and each of which had previously had a high frequency of misconceptions exhibited by students (Swedosh, 1996). Some of the questions were similar to those in the list of questions provided in 'Algebraic Atrocities' (Margulies, 1993, p. 41). Each of the questions on the test was designed so that if a student had a particular misconception, this fact would be apparent when the response of the student to that question was considered.

The test was first administered to both groups in mid-March, a couple of weeks into the semester. Students were given ten minutes to complete the test. This was found to be ample for students to complete the test and did not impinge too greatly on the lecturer's class time. Every student's test was carefully examined on a question by question basis to gain information on any misconceptions which had been exhibited. The number of students who had attempted each question, how many had answered the question correctly, how many had exhibited a misconception, and how many had given another wrong answer were recorded so that comparisons could be made later.

During the semester, both Introductory Mathematics classes met incidentally the concepts and techniques embodied in questions two to seven but did not meet anything relevant to questions one, eight or nine. Questions two to seven therefore become the focus of this study. For the six focus questions, both groups were taught the necessary concepts as they arose in the course. The treatment group were additionally taught using the conflict teaching approach in that they were also shown the absurdities which would arise if the 'rule' was interpreted incorrectly. In other words, at the time of needing to understand the concept, in addition to teaching the concept, the conflict teaching approach was used in an attempt to eliminate or reduce the frequencies with which these misconceptions would occur. The fact that the conflict teaching approach was integrated into the regular teaching programme and was taught incrementally, avoids the Hawthorne effect and is educationally more sound. The misconceptions which were exhibited, having previously been identified and discussed in the literature, were specifically targeted for treatment. An example of this is that 37.8% of students exhibited the misconception that given $1/x - 1/b = 1/a$, then $x - b = a$ and therefore $x = a + b$. The students were shown a similar equation to which the result was self-evident, such as $2 + 3 = 5$. Using the misconception shown above leads to the absurd conclusion that $1/2 + 1/3 = 1/5$. The correct method, using a common denominator, was also taught. The rationale is that having seen that the concept, when used in that way, leads to a silly conclusion, the student will avoid that misconception and would be more likely to use the concept correctly and not be distracted by alternate approaches or tempted to try an illegal 'short-cut'. The other three questions on the test would serve as another control. Neither group would meet concepts relating to these questions during the course of this subject. Even so, the exposure to various mathematical topics could indirectly cause some improvement, but the absence of any related instruction should mean that any improvement is comparable if the two groups are comparable.

In mid-May, about nine weeks after the original test, students in both groups were given the same test as they had sat in March. They were not informed that there would be a second test and therefore would not attempt to prepare for it which would have the potential of affecting the results of the second test. Again the test of each student was carefully examined on a question by question basis and the number of students who had attempted each question, how many had answered the question correctly, how many had exhibited a misconception, and how many had given another wrong answer were recorded. A statistical comparison was then able to be made between the frequency of the misconceptions on each question in the first test and the second test for each group.

After the second test, two of the students who made the greatest improvement were contacted and asked to come in and discuss their performance in the tests. These interviews will be discussed in the results section of this paper.

THE TEST

The nine questions in the test are shown below. The most common misconception(s) are shown to the right of each question :

Simplify expressions 1-3 as fully as possible.

1.	$100!/98!$	$2!$
2.	$3^x \times 3^x$	9^{2x}; 3^{x^2}
3.	$2^x + 2^x$	4^x

Solve for x equations 4 - 7:

4.	$x^2 = 81$	$x = 9$
5	$x^2 - 4x = 0$	$x = 4$
6.	$x^2 = x$	$x = 1$
7.	$1/x - 1/b = 1/a$	$x = a + b$
8.	Solve for $x : 2x + 4 < 5x + 10$	$x < -2$; $x > 2$
9.	Factorise $(2x + y)^2 - x^2$	$3x^2 + 4xy + y^2$

THE TEACHING STRATEGY

The concepts targeted were those required to correctly solve the focus questions. The conflict teaching approach was used on questions two to seven for the treatment group. It was hoped that by demonstrating to the students that if this misconception was employed it would lead to a patently ridiculous conclusion, that as a result, students having this conception would be willing to discard it and replace it with the correct concept. Using this method, it was anticipated (based on Swedosh and Clark (1997)) that the frequency of misconceptions exhibited in the first test would be reduced.

The examples used to show the absurdity caused by the misconception and also to demonstrate the correct concept in each case were mainly numerical. They were slightly different to the questions posed on the test (the numbers were changed slightly, etc.). By doing this, it was thought to be unlikely that students would remember the examples they had been shown and would need to use each concept correctly to answer the respective question. Some of the examples which were shown to the students to demonstrate either the correct concept or the absurdity caused by the damaged concept are shown below.

$2^3 \times 2^3 = 8 \times 8 = 64 = 2^6 = 4^3$; $2^3 \times 2^3 \neq 2^9$; $2^3 \times 2^3 \neq 4^6$; $2^3 \times 2^3 \neq 4^9$

$2^3 + 2^3 = 8 + 8 = 16$; $4^3 = 4 \times 4 \times 4 = 64 \neq 16$

$x^2 = 16, x^2 - 16 = 0, (x-4)(x+4) = 0, x = \pm 4$ also, for a quadratic, we generally expect two solutions (one may be repeated):

$ax^2 + bx + c = 0$; $x = (-b \pm \sqrt{(b^2 - 4ac)})/2a$

$2 + 3 = 5$; $1/2 + 1/3 \neq 1/5$

RESULTS

In this section, information will be provided about the responses to the six focus questions and confidence intervals will be used to determine whether there was a statistically significant difference between the change in the proportion of misconceptions made by the treatment group and that of the control group. In fact, 95% confidence intervals were calculated for the differences in the population proportions of misconceptions between the two groups. If such an interval includes zero, the difference between the groups is not significant. The formula for the confidence intervals is

$$p_1 - p_2 \pm 1.96\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$$

where p_i is the proportion of misconceptions exhibited by group ***i*** and n_i is the number of subjects in group ***i***. The same statistical technique was used on the three questions which were not met in the course, and none of the differences was found to be significant.

Tables 2 and 3 show the frequency of each category of answer given in the first and second tests for the control group and the treatment group respectively. There were 50 students in the control group and 90 students in the treatment group. 'Av. per st.' is the average per student.

Table 2
The Control Group

Question	Correct		Misconception		Other wrong		No attempt	
	Before	After	Before	After	Before	After	Before	After
2	27	27	18	21	2	2	3	0
3	8	4	30	42	5	3	7	1
4	0	18	48	30	1	2	1	0
5	24	30	13	14	12	6	1	0
6	16	23	23	16	8	9	3	2
7	9	9	19	17	15	15	7	9
Total	84	111	151	140	43	37	22	12
Av. per st.	1.68	2.22	3.02	2.80	0.86	0.74	0.44	0.24

Table 3
The Treatment Group

Question	Correct		Misconception		Other wrong		No attempt	
	Before	After	Before	After	Before	After	Before	After
2	48	77	32	9	8	4	2	0
3	7	27	60	48	14	13	9	2
4	6	49	84	41	0	0	0	0
5	51	73	24	12	15	4	0	1
6	34	45	46	29	7	16	3	0
7	9	19	34	24	27	32	20	15
Total	155	290	280	163	71	69	34	18
Av. per st.	1.72	3.22	3.11	1.81	0.79	0.77	0.38	0.20

Comparing the results from the first test, we see that the two groups are very similar: an average of 1.68 correct compared with 1.72, 3.02 misconceptions compared with 3.11, and so on. In fact, none of the differences between the groups was found to be significant on any question or on the total. It is interesting to note that the average frequency of 'other wrong answers' given (which were not specifically targeted) was much the same for both groups and showed little change from before treatment to after.

Table 4 shows the percentages of each category of answer given in the second test for each group. In Table 4 'Q' is the question number. The last three columns show the results of the confidence interval which compares the proportion of misconceptions exhibited by each group. The first two of these columns give an estimate at the 95% confidence level for the interval in which the difference between the population proportions would lie. As previously stated, if the interval includes zero, the difference is not significant. In the last column, * indicates those differences which are significant.

Table 4

Q	Of those who attempted the question on the Test 2						Significance test (misconceptions)		Sig. Diff.
	% Correct		% Misconception		% Other error		Test 2		
	Control	Treat.	Control	Treat.	Control	Treat.			
2	54.0	85.6	42.0	10.0	4.0	4.4	0.1698	0.4702	*
3	8.2	30.7	85.7	54.5	6.1	14.8	0.1703	0.4531	*
4	36.0	54.4	60.0	45.6	4.0	0.0	-0.0259	0.3148	
5	60.0	82.0	28.0	13.5	12.0	4.5	0.0021	0.2882	*
6	47.9	50.0	33.3	32.2	18.8	17.8	-0.1514	0.1736	
7	22.0	25.3	41.5	32.0	36.6	42.7	-0.0725	0.2618	
Av.	38.0	54.7	48.4	31.3	13.6	14.0	0.0027	0.3396	*

The results indicate that there is a significant difference between the overall proportions of misconceptions exhibited by the two groups. The proportions of misconceptions for the control group dropped from 54.3% to 48.4% from the first to the second test while those of the treatment group dropped from 55.3% to 31.3%. There is a corresponding increase in the proportion of correct answers given by the treatment group relative to the control group. Five of the six questions show an appreciable difference between the two groups, with three of these differences significant. Question six was the exception.

Soon after the second test had been administered, two of the students from the treatment group who made the greatest improvement from the first test to the second were contacted and invited to an interview to discuss their performance in the tests. The two students were interviewed separately. Both students offered similar explanations for their improvement in that each stated, without prompting, that there were occasions in the second test when they were about to write down an answer, but they realised that using the method which they had, would lead to a silly answer. Both mentioned that they had come to this decision by remembering what they had been shown in class in terms of the 'silliness' which could result from some strategies, and by substituting in numbers to see whether the answer might be reasonable. They then considered the need for an alternate method and remembered the correct one.

CONCLUSIONS

This study set out to ascertain how generalisable were the benefits of the teaching strategy used, based on the notion of cognitive conflict, when considering students of different ability levels. The strategy had previously been found to be extremely effective with very able students, and it had been found that the benefit persisted over time. To establish that this method was useful beyond the scope of a very bright and somewhat atypical group and that it could be successfully employed when dealing with an average group of students, would make it a much more useful strategy. For this reason, it was considered to be very important to learn whether the success of the strategy could be generalised to that extent.

The experiment conducted provides strong evidence that the conflict teaching strategy can be successfully employed to significantly decrease the frequency of misconceptions exhibited by average students. The specifically targeted misconceptions were decreased for the treatment group by a significantly greater amount than for the control group, whereas the 'other wrong answers' (which were not specifically targeted) were not, and neither was the frequency of misconceptions on the three questions which were not taught during the course. The major benefit to be gained from incorporating this strategy, which is extremely simple to implement, into one's teaching, is that not only is it likely that fewer students will have these misconceptions, but, as a result of this, many will directly improve their chances of being successful in their future studies of mathematics. It is suggested that the use of this strategy, especially with regard to the concepts herein, is particularly applicable to senior secondary mathematics as well as to beginning tertiary mathematics.

REFERENCES

Australian Education Council. (1990). *A National Statement on Mathematics for Australian Schools.* Carlton, Victoria: Curriculum Corporation.

Bell, A.W. (1982). Diagnosing students misconceptions. *The Australian Mathematics Teacher, 38*, 1, 6-10.

Blyth, B., & Calegari, J. (1985). The Blackburn Report: Nightmare or Challenge? From a Tertiary Perspective. In P. Sullivan (Ed.), *Mathematics — Curriculum, Teaching & Learning* (pp 309-314). Melbourne: The Mathematical Association of Victoria.

Davis, R.B. (1984). *Learning Mathematics : The cognitive science approach to mathematics education*, London : Croom Helm.

Farrell, M.A. (1992). Implementing the professional standards for teaching mathematics: Learning from your students. *The Mathematics Teacher, 85* (8), 656-659.

Margulies, S. (1993). Tips for beginners : Algebraic atrocities. *Mathematics Teacher, 86* (1), 40-41.

Ministry of Education (Victoria). (1984). *Ministerial Paper Number 6 — Curriculum development and planning in Victoria.* Melbourne: Government Printer.

Perso, T. (1992). Making the most of errors. *The Australian Mathematics Teacher, 48* (2), 12-14.

Stavy, R., & Berkovitz, B. (1980). Cognitive conflict as a basis for teaching quantitative aspects of the concept of temperature. *Science Education , 64*, 672-692.

Strauss, S. (1972). Inducing cognitive development and learning: A review of short-term training experiments: 1) The organismic-developmental approach. *Cognition, 1*, 329-357.

Swan, M. (1983). *Teaching decimal place value. A comparative study of "conflict" and "positive only" approaches.* Nottingham, England: University of Nottingham, Shell Centre for Mathematical Education.

Swedosh, P. (1996). Mathematical misconceptions commonly exhibited by entering tertiary mathematics students. In P. Clarkson (Ed.), *Technology in Mathematics Education* (pp. 534-541). Melbourne: Mathematics Education Research Group of Australasia.

Swedosh, P., & Clark, J. (1997). Mathematical Misconceptions — Can We Eliminate Them? In F.Biddulf & K. Carr (Eds.), *People in Mathematics Education* (2), (pp. 492-499). Waikato: Mathematics Education Research Group of Australasia.

Swedosh, P., & Clark, J. (1998). Mathematical Misconceptions — We have an Effective Method for Reducing Their Incidence But Will the Improvement Persist? In C. Kanes, M. Goos, E. Warren (Eds.), *Teaching Mathematics in New Times* (2), (pp. 588-595). Broadbeach: Mathematics Education Research Group of Australasia.

Tirosh, D. (1990). Inconsistencies in students mathematical constructs. *Focus on Learning Problems in Mathematics, 12*, (3-4), 111-129.

Victorian Government, (1987). *Victoria The Next Decade.* Melbourne: Government Printer.

Vinner, S. (1990). Inconsistencies: Their causes and function in learning mathematics. *Focus on Learning Problems in Mathematics, 12*, (3-4), 85-98.

Worley, A.K. (1993, June). *Pre-calculus skills — The extent of the problem.* The Australian Bridging Mathematics Conference, Brisbane.

Acknowledgements

I would like to acknowledge the cooperation of Frank Barrington and Angie Byrne and the advice provided by David Clarke. Their efforts are sincerely appreciated.